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BETHE ANSATZ AND CLASSICAL HIROTA EQUATIONS [1]

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A brief non-technical review of the recent study [1] of classical integrable structures in quantum integrable systems is given. It is explained how to identify the standard objects of quantum integrable systems (transfer matrices, Baxter's Q-operators, etc) with elements of classical non-linear integrable difference equations (τ-functions, Baker-Akhiezer functions, etc). The nested Bethe ansatz equations for A_{k-1}-type models emerge as discrete time equations of motion for zeros of classical τ-functions and Baker-Akhiezer functions. The connection with discrete time Ruijsenaars-Schneider system of particles is discussed.

At present it is argued that classical and quantum integrable models have much deeper interrelations than any kind of a naive "classical limit". In [1], a particular aspect of this phenomenon has been analysed in detail: Bethe equations, which are usually considered as a tool inherent to quantum integrability, arise naturally as a result of solving entirely *classical* non-linear discrete time integrable equations. This suggests an intriguing connection between integrable quantum field theories and classical soliton equations in discrete time.

Here we outline main ideas of the paper [1] omitting all technical details. To save the space, the material is organized as a "quantum - classical dictionary" supplied with brief comments.

1. Eigenvalues of quantum transfer matrices = (classical) τ-functions. Due to the Yang-Baxter equation the transfer matrices commute for all values of the spectral parameters in the auxiliary space (AS)[2]: $[T_{\mathcal{A}}(u), T_{\mathcal{A}'}(u')] = 0$. This property allows one to diagonalize them simultaneously. From now on we use this diagonal representation. The identification with τ-function is justified in the next item.

2. Fusion rules = Hirota's difference equation. The fusion procedure in the AS gives rise to a family of commuting transfer matrices $T_{\mathcal{A}}(u)$ with the same quantum space. They obey a number of fusion relations [4] which can be recast into the model-independent bilinear form [5]. Let $T_s^a(u)$ be the transfer matrix for the rectangular Young diagram of length s and height a, then it holds

$$T_s^a(u+1)T_s^a(u-1) - T_{s+1}^a(u)T_{s-1}^a(u) = T_s^{a+1}(u)T_s^{a-1}(u)\,. \qquad (1)$$

Remarkably, this equation coincides with Hirota's bilinear difference equation (HBDE) [6] which is known to unify the majority of soliton equations, both discrete and continuous.

Fusion of more complicated representations in the AS is described by higher representatives of the hierarchy of HBDE-like equations [7]. For example, consider Young diagrams consisting of two rectangular blocks (i.e. with a_1 lines of length $s_1 + s_2$ and the rest a_2 lines of length s_1) and let $T_{s_1,s_2}^{a_1,a_2}(u)$ be the corresponding transfer matrix. Then it holds

$$\begin{aligned} & T_{s_1,s_2}^{a_1,a_2-1}(u)T_{s_1-1,s_2-1}^{a_1,a_2+1}(u) + T_{s_1,s_2+1}^{a_1-1,a_2-1}(u)T_{s_1,s_2-1}^{a_1+1,a_2+1}(u) \\ + \;& T_{s_1+1,s_2}^{a_1-1,a_2}(u-1)T_{s_1-1,s_2}^{a_1+1,a_2}(u+1) \\ = \;& T_{s_1+1,s_2}^{a_1,a_2-1}(u-1)T_{s_1-1,s_2}^{a_1,a_2+1}(u+1) + T_{s_1,s_2+1}^{a_1-1,a_2}(u-1)T_{s_1,s_2-1}^{a_1+1,a_2}(u+1)\,. \end{aligned} \qquad (2)$$

[1]Talk presented at the II Sakharov International Conference

[2]Here and below we use the notions and terminology of the quantum inverse scattering method, see e.g. [2].

In what follows we consider eq. (1) only.

3. Specifying a particular quantum model = imposing particular boundary and analytic conditions in HBDE.

For models associated to A_{k-1}-type quantum algebras these conditions are:

$$\begin{aligned} &T_s^a(u)=0 \quad \text{as } \ a<0 \ \text{ or } \ a>k;\\ &T_s^a(u)=0 \quad \text{for any } \ -k<s<0, \ \ 0<a<k\,. \end{aligned} \tag{3}$$

It follows from (1), (3) that $T_s^0(u)$ and $T_s^k(u)$ factorize into a product of functions of $u+s$ and $u-s$. We adopt the normalization in which $T_s^0(u)=\phi(u+s)$, $T_s^k(u)=\phi(u-s-k)$, where the function $\phi(u)$ carries information about the particular quantum model.

Another important condition (which follows, eventually, from the Yang-Baxter equation) is that $T_s^a(u)$ for models with elliptic R-matrices has to be an elliptic polynomial in the spectral parameter u. (By elliptic polynomial we mean essentially a finite product of Weierstrass σ-functions.) For models with rational R-matrix it degenerates to a usual polynomial in u.

4. "Step" of nested Bethe ansatz = Bäcklund transformation. Quantum integrable models with internal degrees of freedom can be solved by the nested (hierarchical) Bethe ansatz method. The method consists essentially in integration over a part of degrees of freedom by an ansatz of Bethe type, the effective hamiltonian being again integrable. Repeating this step several times, one reduces the model to an integrable model without internal degrees of freedom which is solved by the usual Bethe ansatz.

The classical face of this scheme is Bäcklund transformation, i.e. passing from solutions to the non-linear equation to (properly normalized) solutions of the auxiliary linear problems (ALP), which satisfy the same non-linear equation. The ALP for HBDE have the form [6]

$$\begin{aligned} &T_{s+1}^{a+1}(u)F^a(s,u)-T_s^{a+1}(u-1)F^a(s+1,u+1)=T_s^a(u)F^{a+1}(s+1,u)\,,\\ &T_{s+1}^{a}(u-1)F^a(s,u)-T_s^{a}(u)F^a(s+1,u-1)=T_s^{a+1}(u-1)F^{a-1}(s+1,u)\,, \end{aligned} \tag{4}$$

Eq. (1) is the compatibility condition for (4). Due to the symmetry between T and F [8], the latter satisfies *the same* non-linear equation (1), so the transition $T\to F$ is a Bäcklund transformation. The important point is that the boundary condition for $F^a(s,u)$ is the same as in (3), the only change being a reduction of the Dynkin graph: $k\to k-1$. In other words, the number of non-zero functions gets reduced by 1. Using this property, one can successively reduce the A_{k-1}-problem up to A_1.

5. Eigenvalues of Baxter's Q-operators = properly normalized solutions to the ALP (Baker-Akhiezer functions). To elaborate the chain of Bäcklund transformations, let $t=0,\,1,\ldots,k$ mark steps of the flow $A_{k-1}\to A_1$ and let $F_{t+1}^a(s,u)$ be a solution to the ALP at $(k-t)$-th step (in this notation $F_k^a(s,u)=T_s^a(u)$) such that $F_t^a(s,u)=0$ as $a<0$ or $a>t$. Due to this condition F_t^0 and F_t^t are "chiral" functions, i.e.

$$F_t^0(s,u)=Q_t(u+s)\,,\qquad F_t^t(s,u)=Q_t(u-s-t)\,, \tag{5}$$

where $Q_t(u)$ are some functions playing a distinguished role in what follows since they can be identified with generalized Baxter's Q-operators in the diagonal representation (see below). Here is an example of this array of τ-functions for the A_2-case ($k=3$):

$$\begin{array}{cccccc} 0 & 1 & 0 & & & \\ 0 & Q_1(u+s) & Q_1(u-s-1) & 0 & & \\ 0 & Q_2(u+s) & F_2^1(s,u) & Q_2(u-s-2) & 0 & \\ 0 & Q_3(u+s) & F_3^1(s,u) & F_3^2(s,u) & Q_3(u-s-3) & 0 \end{array} \tag{6}$$

(in this case $Q_3(u) = \phi(u)$).

6. Nested Bethe ansatz equations = Calogero-type models in discrete time. It follows from (4) that $\tau^{t,a}(u) = F^a_{k-t}(u+a, u)$ satisfies the bilinear equation

$$\tau^{t+1,a}(u)\tau^{t,a+1}(u) - \tau^{t,a}(u)\tau^{t+1,a+1}(u) = \tau^{t+1,a}(u+1)\tau^{t,a+1}(u-1) \tag{7}$$

which is HBDE in "light cone" variables. At the same time $\tau^{t,0}(u) = Q_{k-t}(2u)$. Let u^t_j be zeros of $Q_t(u)$: $Q_t(u^t_j) = 0$. For models with elliptic R-matrices $Q_t(u)$ should be elliptic polynomials in u. This condition leads to a number of constraints for u^t_j which can be derived using the technique developed in [9] for elliptic solutions to the KP equation. These constraints are nothing else than the nested Bethe ansatz equations:

$$\frac{Q_{t-1}(u^t_j+2)Q_t(u^t_j-2)Q_{t+1}(u^t_j)}{Q_{t-1}(u^t_j)Q_t(u^t_j+2)Q_{t+1}(u^t_j-2)} = -1 \tag{8}$$

(with the boundary condition $Q_0(u) = 1$, $Q_k(u) = \phi(u)$). These equations can be understood as "equations of motions" for zeros of $Q_t(u)$ in discrete time t (level of the Bethe ansatz which runs over the Dynkin graph). The analogy with elliptic solutions of the KP equation suggests to call them the discrete time analogue of the Ruijsenaars-Schneider (RS) system of particles (see also [10]). Taking the continuum limit in t (provided the number of zeros $M_t = M$ of $Q_t(u)$ in a fundamental domain does not depend on t), one can verify that this system of equations does yield equations of motion for the RS system [11] with M particles.

However, integrable systems of particles in discrete time have a richer structure than their continuous counterparts. In particular, the total number of particles may depend on the discrete time. Such a phenomenon is possible in continuous time models only for singular solutions, when particles can move to infinity or merge to another within a finite period of time. Remarkably, this appears to be the case for the solutions to eq. (8) corresponding to eigenstates of quantum models. It is known that the number of excitations at t-th level of the nested Bethe ansatz solution does depend on t. In other words, the number of "particles" in the associated discrete time RS system is not conserved. At the same time the numbers M_t may not be arbitrary. It can be shown that for models with elliptic R-matrices in case of general position $M_t = (N/k)t$, where N is the number of sites of the lattice (degree of the elliptic polynomial $\phi(u)$). In trigonometric and rational cases the conditions on M_t become less restrictive but still they may not be equal to each other.

At last we should indicate how to identify our Q_t's with Q_t's from the usual nested Bethe ansatz solution. This is achieved by the following factorization formula which allows one to express $T^a_s(u)$ in terms of Q_t's:

$$\sum_{a=0}^{k}(-1)^{a-k}\frac{T^a_1(u+a-1)}{\phi(u-2)}e^{2a\partial_u} =$$
$$=\left(e^{2\partial_u} - \frac{Q_k(u)Q_{k-1}(u-2)}{Q_k(u-2)Q_{k-1}(u)}\right)\ldots\left(e^{2\partial_u} - \frac{Q_2(u)Q_1(u-2)}{Q_2(u-2)Q_1(u)}\right)\left(e^{2\partial_u} - \frac{Q_1(u)}{Q_1(u-2)}\right), \tag{9}$$

where the shift operator e^{∂_u} acts as usual: $e^{\partial_u}f(u) = f(u+1)$. For the proof see ref. [1]. This formula coincides with the one known in the literature (see e.g. [12]). The left hand side is known as the generating function for $T^a_1(u)$.

There is no doubt that this dictionary can be continued to include not only spectral properties of quantum system on finite lattices (which just correspond to elliptic solutions of HBDE) but dynamical properties and correlation functions as well. Perhaps this amounts to considering solutions to the same Hirota equation of a more complicated type. Another intriguing problem is to extend this dictionary to off-diagonal elements of quantum monodromy matrices and quantum L-operators and identify them with some objects in hierarchies of classical integrable equations.

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References

[1] I.Krichever, O.Lipan, P.Wiegmann and A.Zabrodin, preprint ESI 330 (1996), hep-th/9604080.

[2] L.D.Faddeev and L.A.Takhtadjan, Uspekhi Mat. Nauk **34:5** (1979) 13; P.P.Kulish and E.K.Sklyanin, Lecture Notes in Physics **151** 61, Springer, 1982.

[3] L.D.Faddeev and L.A.Takhtadjan, *Quantum inverse scattering method and the XYZ Heisenberg model*, Uspekhi Mat. Nauk **34:5** (1979) 13-63.

[4] P.P.Kulish and N.Yu.Reshetikhin, Zap. Nauchn. Sem. LOMI **120** (1982) 92 (in russian).

[5] A.Klümper and P.Pearce, Physica **A183** (1992) 304; A.Kuniba, T.Nakanishi and J.Suzuki, Int. Journ. Mod. Phys. **A9** (1994) 5215.

[6] R.Hirota, Journ. Phys. Soc. Japan **50** (1981) 3785.

[7] E.Date, M.Jimbo and T.Miwa, Journ. Phys. Soc. Japan **53** (1982) 4116.

[8] S.Saito and N.Saitoh, Phys. Lett. **A120** (1987) 322.

[9] I.M.Krichever, Func. Anal. App **14** (1980), n 4, 282.

[10] F.Nijhof, O.Ragnisco and V.Kuznetsov, Commun. Math. Phys. **176** (1996) 681.

[11] S.N.M.Ruijsenaars and H.Schneider, Ann. Phys. (NY) **170** (1996) 681.

[12] V.Bazhanov and N.Reshetikhin, Journ. Phys. **A23** (1990) 1477.